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Singularity of Bivariate Interpolation

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We study a question of G. Lorentz and R. Lorentz about the singularity in the bivariate interpolation. Infinitely many new singular matrices are constructed which correspond to the Hermite interpolation with equal multiplicities. We find a geometrical interpretation which describes all singular matrices given by our method. Finally, simple sufficient conditions for singularity in the general case are given.

1. INTRODUCTION

We consider the classical interpolation problem in $\mathbb{R}^s$. Only notational difficulties separate the cases $s>2$ and $s=2$ and all our results can be extended to $\mathbb{R}^s$, $s>2$. Henceforth, we suppose that $s=2$.

Denote by $\Pi_n(\mathbb{R}^2)$ the set of all bivariate algebraic polynomials of the form $P(x, y)=\sum_{(s,m)\in S} a_{sm}(x^s/s!)(y^m/m!)$, where $\{a_{sm}\}$ are real coefficients and $S:=S_n:=\{(s, m): s+m\leqslant n,\ s, m$ are non-negative integers$\}$.

Let $Z=\{z_i\}_{i=1}^p=\{(x_i, y_i)\}_{i=1}^p$ be a set of distinct points in $\mathbb{R}^2$. Given the numbers $c_{i,j,k}$ and the multiplicities $\{v_i\}_{i=1}^p$ we study the interpolation problem

$$\left.\frac{\partial^{j+k}P}{\partial x^j\,\partial y^k}\right|_{z_i}=c_{i,j,k} \qquad \text{for} \quad j+k\leqslant v_i, \quad i=1, ..., p, \tag{1}$$

where $P(x, y)\in\Pi_n(\mathbb{R}^2)$.

The conditions (1) are usually described by the corresponding incidence matrices $E_{v_i}=(e_{i,j,k})_{(j,k)\in S}$, where

$$e_{i,j,k}=\begin{cases}0 & \text{if}\quad j+k>v_i\\ 1 & \text{if}\quad j+k\leqslant v_i,\end{cases}$$

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Following [1] we denote by E the collection of all matrices $\{E_{\nu_i}\}$, i.e.,

$$E = E_{\nu_1} \oplus \cdots \oplus E_{\nu_p}. \tag{2}$$

Henceforth we suppose that the matrix E is normal, i.e.,

$$\sum_{i=1}^{p} \binom{\nu_i + 2}{2} = \binom{n+2}{2}. \tag{3}$$

Denote by $D(E, Z)$ the determinant of the system (1) with respect to the unknowns $\{a_{sm}\}$.

The matrix E is called *singular* if $D(E, Z) \equiv 0$ for all Z. E is *almost regular* otherwise.

G. Lorentz and R. Lorentz [2] posed the problem of complete characterization of the singular matrices in terms of the multiplicities $\{\nu_i\}_{i=1}^{p}$. Despite the efforts expended to answer this question there are only a few examples known of singular matrices (see [1, 2]). We give in this paper general sufficient conditions for singularity. Using this results, we completely characterize the interpolation problem when $2 \leqslant p \leqslant 9$.

2. A Method for Singularity

We give first some preliminaries.

We say that the polynomial $P(x, y)$ has zero of multiplicity $\alpha + 1 \geqslant 0$ at the point $(x_0, y_0) = z_0 \in \mathbb{R}^2$ if

$$\left. \frac{\partial^{j+k} P}{\partial x^j \, \partial y^k} \right|_{z_0} = 0 \qquad \text{for} \quad j + k \leqslant \alpha.$$

In case $\alpha = -1$ we stipulate that $P(x_0, y_0) \neq 0$.

The next lemma follows easily from the Leibnitz formula for differentiation of products.

Lemma 1. *Let the polynomials $Q(x, y)$ and $R(x, y)$ have zeros of multiplicities $\alpha + 1$ and $\beta + 1$ at $z_0 \in \mathbb{R}^2$, respectively. Then $Q(x, y) R(x, y)$ has zero of multiplicity $\alpha + \beta + 2$ at z_0.*

Further, we make use of the following lemma.

Lemma 2. *Let $Z = \{z_j\}_{j=1}^{p}$ be a set of distinct points in $\mathbb{R}^2$ and the integers $r \geqslant 1$ and $\alpha_j \geqslant 0$, $j = 1, ..., p$ satisfy the requirement*

$$\sum_{j=1}^{p} \binom{\alpha_j + 2}{2} + 1 \leqslant \binom{r+2}{2}.$$

Then there exists a nontrivial polynomial $Q(x, y) \in \Pi_r(\mathbb{R}^2)$ *which has zeros of multiplicities* $\alpha_j + 1$ *at the points* z_j.

The next observation produces sufficient conditions for singularity.

THEOREM 1. *Let E be a normal matrix of type* (2). *If there exist integer numbers* q, $\{r_i\}_{i=1}^q$, $\{\alpha_{ij}\}_{i=1}^q{}_{j=1}^p$, $2 \leqslant q \leqslant n$, $r_i \geqslant 1$, $\alpha_{ij} \geqslant -1$, *and*

$$\sum_{i=1}^{q} \alpha_{ij} \geqslant \nu_j + 1 - q \qquad \text{for} \quad j = 1, ..., p, \tag{4}$$

$$\sum_{i=1}^{q} r_i \leqslant n, \tag{5}$$

$$\sum_{j=1}^{p} \binom{\alpha_{ij}+2}{2} + 1 \leqslant \binom{r_i+2}{2} \qquad \text{for} \quad i = 1, ..., q, \tag{6}$$

then the matrix E is singular.

Proof. Let $Z = \{z_j\}_{j=1}^p$ be an arbitrary set of distinct points in $\mathbb{R}^2$. By Lemma 2 and (6) for each $i = 1, ..., q$ there exists a nontrivial polynomial $Q_i(x, y) \in \Pi_{r_i}(\mathbb{R}^2)$ which vanishes at the points $\{z_j\}_{j=1}^p$ of multiplicities $\alpha_{ij} + 1$, respectively. Consider the polynomial $P(x, y) = Q_1(x, y) Q_2(x, y) \cdots Q_q(x, y)$. Clearly, $P(x, y)$ is nontrivial and $P(x, y) \in \Pi_n(\mathbb{R}^2)$. An application of Lemma 1 and (4) shows that $P(x, y)$ has a zero of multiplicity $\sum_{i=1}^q \alpha_{ij} + q \geqslant \nu_j + 1$ at z_j. Thus $D(E, Z) = 0$ for any Z. The proof is completed. ∎

The matrix E is said to be *perfect singular* (*p-singular*) if there exist integer numbers q, $\{r_i\}_{i=1}^q$, $\{\alpha_{ij}\}_{i=1}^q{}_{j=1}^p$ which satisfy the assumptions of Theorem 1.

Without loss of generality we shall consider that (4) and (5) are equalities. Note that the condition (6) may be written in the form

$$\sum_{j=1}^{p} \left(\alpha_{ij} + \frac{3}{2}\right)^2 \leqslant \left(r_i + \frac{3}{2}\right)^2 + \frac{p-9}{4}, \qquad i = 1, ..., q. \tag{7}$$

Next we give a geometrical interpretation of our method which we shall use later. Denote by $K_i \subset \mathbb{R}^p$ the closed ball with center $T_i(\{-3/2, p\}) \in \mathbb{R}^p$ and radius $R_i = \sqrt{(r_i + 3/2)^2 + (p-9)/4}$, where $\{m, p\}$ means that m is repeated p times.

We say that $A_1 + \cdots + A_m$ is algebraic sum of $A_i \subset \mathbb{R}^p$, $i = 1, ..., m$ if $A_1 + \cdots + A_m := \{a_1 + \cdots + a_m : a_1 \in A_1, ..., a_m \in A_m\}$.

The following simple lemma can easily be verified.

LEMMA 3. *The set* $K = K_1 + K_2 + \cdots + K_q$ *is a closed ball with center* $(\{-\frac{3}{2}q, p\}) \in \mathbb{R}^p$ *and radius* $R = R_1 + \cdots + R_q$.

Thus using representations (7), we obtain

COROLLARY 1. *The matrix E of type* (2) *is p-singular if and only if there exist natural numbers* $q \geq 2$, $\{r_i\}_{i=1}^q$, $\sum_{i=1}^q r_i = n$ *such that the point* $(v_1 + 1 - q, ..., v_p + 1 - q)$ *can be represented as a sum of the points with integer coordinates* ≥ -1, $z_1 + z_2 + \cdots + z_q$, $z_i \in K_i \subset \mathbb{R}^p$.

The next theorem shows that if E is p-singular then we may choose $q = 2$ in Theorem 1.

THEOREM 2. *Let E be a normal p-singular matrix of type* (2). *Then there exist integers* $r_i \geq 1$, $i = 1, 2$, $\alpha_j, \beta_j \geq -1$, $j = 1, ..., p$ *such that*

$$\alpha_j + \beta_j = v_j - 1, \qquad j = 1, ..., p,$$

$$r_1 + r_2 = n,$$

$$\sum_{j=1}^{p} \binom{\alpha_j + 2}{2} + 1 \leq \binom{r_1 + 2}{2}, \qquad \sum_{j=1}^{p} \binom{\beta_j + 2}{2} + 1 \leq \binom{r_2 + 2}{2}.$$

Proof. Since E is p-singular, there exists some integer numbers $\ddot{A} = (q, r_i, \alpha_{ij})$ satisfying the requirement of Theorem 1. Our goal is to show that q can be chosen to be 2. We apply induction on q.

Assume that $q = m > 2$ in $\ddot{A}$. Now we find numbers $\ddot{A}$ with smaller q. Let $Z = \{z_j\}_{j=1}^p$ be an arbitrary set of distinct points in $\mathbb{R}^2$. We saw in the proof of Theorem 1 that there exist nontrivial polynomials $Q_i(x, y) \in \Pi_{r_i}(\mathbb{R}^2)$, $i = 1, ..., m$, which have zeros at z_j of multiplicity $\alpha_{ij} + 1$. Moreover z_j is a zero of $P(x, y) = Q_1(x, y) \cdots Q_m(x, y)$ of multiplicity $v_j + 1$.

Let $i \neq k$, $1 \leq i, k \leq m$. It follows from Lemma 1 that $Q_i(x, y) Q_k(x, y)$ has a zero of multiplicity $\alpha_{ij} + \alpha_{kj} + 2$ at the z_j. Note that if

$$\sum_{j=1}^{p} \binom{\alpha_{ij} + \alpha_{kj} + 3}{2} + 1 \leq \binom{r_i + r_k + 2}{2}$$

for some i, k the problem is reduced to the case $q = m - 1$ because the numbers $m - 1$, $r_1, ..., r_{i-1}$, $r_{i+1}, ..., r_{k-1}$, $r_{k+1}, ..., r_m$, r', $\alpha_{1j}, ..., \alpha_{i-1j}$, $\alpha_{i+1j}, ..., \alpha_{k-1j}$, $\alpha_{k+1j}, ..., \alpha_{mj}$, α'_j, with $\alpha'_j = \alpha_{ij} + \alpha_{kj} + 1$, $r' = r_i + r_k$ satisfy the assumptions of Theorem 1. So let us suppose that

$$\sum_{j=1}^{p} \binom{\alpha_{ij} + \alpha_{kj} + 3}{2} \geq \binom{r_i + r_k + 2}{2}, \qquad \text{for every} \quad i \neq k,\ 1 \leq i, k \leq m. \quad (8)$$

For fixed i, consider the polynomial $Q_1 \cdots Q_{i-1} Q_{i+1} \cdots Q_m$ of degree $n - r_i$. By Lemma 1 it has zeros of multiplicities $v_j - \alpha_{ij}$ at the points z_j. If

$\sum_{j=1}^{p}\binom{v_j+1-\alpha_{ij}}{2}+1\leqslant\binom{n-r_i+2}{2}$, we set $q=2$, $r'=r_i$, $r''=n-r_i$, $\alpha_j=\alpha_{ij}$, $\beta_j=v_j-1-\alpha_{ij}$ and the problem is reduced to $q=2$. So, let us suppose that

$$\sum_{j=1}^{p}\binom{v_j+1-\alpha_{ij}}{2}\geqslant\binom{n-r_i+2}{2},\qquad \text{for every}\quad 1\leqslant i\leqslant m. \tag{9}$$

It follows from condition (6) and (8) that

$$\sum_{j=1}^{p}\left(\binom{\alpha_{ij}+\alpha_{kj}+3}{2}-\binom{\alpha_{ij}+2}{2}-\binom{\alpha_{kj}+2}{2}\right)$$
$$\geqslant\binom{r_i+r_k+2}{2}-\binom{r_i+2}{2}-\binom{r_k+2}{2}+2$$

for each pair i, k, $i\neq k$. After some straightforward transformations the above inequality takes the form

$$\sum_{j=1}^{p}(\alpha_{ij}+1)(\alpha_{kj}+1)\geqslant r_ir_k+1,\qquad \text{for every}\quad i\neq k. \tag{10}$$

Similarly, from (3) and (9) we get

$$\sum_{j=1}^{p}\left(\binom{v_j+2}{2}-\binom{v_j+1-\alpha_{ij}}{2}\right)\leqslant\binom{n+2}{2}-\binom{n-r_i+2}{2}$$

and hence

$$\sum_{j=1}^{p}(\alpha_{ij}+1)\left(v_j+1-\frac{\alpha_{ij}}{2}\right)\leqslant r_i\left(n-\frac{r_i-3}{2}\right),\qquad 1\leqslant i\leqslant m. \tag{11}$$

But $\sum_{k=1}^{m}(\alpha_{kj}+1)=v_j+1$ according to (4). Thus (11) becomes

$$\sum_{\substack{k=1\\k\neq i}}^{m}\sum_{j=1}^{p}(\alpha_{ij}+1)(\alpha_{kj}+1)+\sum_{j=1}^{p}\binom{\alpha_{ij}+2}{2}\leqslant r_i\left(n-\frac{r_i-3}{2}\right).$$

Now, using (10) and (5), we obtain

$$\sum_{\substack{k=1\\k\neq i}}^{m}(r_ir_k+1)+\sum_{j=1}^{p}\binom{\alpha_{ij}+2}{2}\leqslant r_i\left(\sum_{\substack{k=1\\k\neq i}}^{m}r_k+\frac{r_i+3}{2}\right)$$

which is reduced to

$$\sum_{j=1}^{p}\binom{\alpha_{ij}+2}{2}+M\leqslant\binom{r_i+2}{2},\qquad 1\leqslant i\leqslant m, \tag{12}$$

where $M = m$. Remark that (12) is stronger than (6) because m stands here instead of 1. Thus repeating this argument again starting with (12) instead of (6) we would get an inequality of type (12) with $M = (m-1)(2m-1)+1$. Now it is clear that this procedure leads to (12) with arbitrary large M, thus to a contradiction. The theorem is proved. ∎

Using Corollary 1 and Theorem 2 we obtain

COROLLARY 2. *The matrix E of type* (2) *is p-singular if and only if there exist natural numbers r_1 and r_2, $r_1 + r_2 = n$ such that the point $M(v_1 - 1, v_2 - 1, ..., v_p - 1)$ can be represented as a sum $X(x_1, ..., x_p) + Y(y_1, ..., y_p)$, $X \in K_1$, $Y \in K_2$, and $x_j \geqslant -1$, $y_j \geqslant -1$ are integers.*

If the matrix E is p-singular then $M \in K = K_1 + K_2$. Thus, from Lemma 3, we derive the following

COROLLARY 3. *If the matrix E is p-singular then there exists a natural number $r_1 \leqslant n-1$ such that*

$$R_1 + R_2 \geqslant \sqrt{\sum_{j=1}^{p} (v_j + 2)^2}, \tag{13}$$

where $r_2 = n - r_1$ and

$$R_i = \sqrt{\left(r_i + \frac{3}{2}\right)^2 + \frac{p-9}{4}}. \tag{14}$$

3. HERMITIAN CASE OF EQUAL MULTIPLICITIES

Suppose here that $v_1 = \cdots = v_p = v$, i.e., the matrices of the type

$$E = E_v \oplus \cdots \oplus E_v - p \text{ times}, \qquad v \geqslant 0 \tag{15}$$

will be considered. We shall find necessary and sufficient conditions for p-singularity and will describe all p-singular matrices with equal multiplicities. The matrix E is supposed to be normal, i.e.,

$$p\binom{v+2}{2} = \binom{n+2}{2}. \tag{16}$$

It is interesting to find the set of non-negative integers (n, v) which satisfy (16). Clearly, for such (n, v),

$$(2n+3)^2 - p(2v+3)^2 = 1 - p. \tag{17}$$

Then set $x=2n+3$, $y=2v+3$. Thus (17) takes the form

$$x^2-py^2=1-p. \tag{18}$$

This is a well known Diophantine equation. Its solutions

$$\begin{aligned} x&=ax'+pby' \\ y&=ay'+bx' \end{aligned} \tag{19}$$

are found on the basis of the solutions (a, b) of the famous Pell's equation (see [4])

$$a^2-pb^2=1, \tag{20}$$

where (x', y') is a particular solution of (18). It is proved in [4] that all integer solutions of (18) are obtained from some finite class of particular solutions (x', y') by formulae (19). On the other hand, the non-negative integer solutions of (20) when $\sqrt{p}$ is not integer are

$$a_m=\frac{1}{2}((a_0+b_0\sqrt{p})^m+(a_0-b_0\sqrt{p})^m),$$

$$b_m=\frac{1}{2\sqrt{p}}((a_0+b_0\sqrt{p})^m-(a_0-b_0\sqrt{p})^m),$$

$m\geqslant 0$, where (a_0, b_0) is the minimal positive solution of (20). Clearly, particular solutions of (18) are $x'=1$, $y'=1$ and $x'=-1$, $y'=1$. Then the following sequences

$$\left| \begin{aligned} x_m&=a_m+pb_m \\ y_m&=a_m+b_m \end{aligned} \right. \qquad \left| \begin{aligned} x'_m&=-a_m+pb_m \\ y'_m&=a_m-b_m \end{aligned} \right. , \qquad m\geqslant 1$$

are solutions of (18). When $p=2$ and $p=3$ the above sequences coincide (see [4]). Then $n=(x-3)/2$, $v=(y-3)/2$ where $x, y\geqslant 3$ are odd integer solutions of (18).

It follows by Corollary 2 that the matrix E of type (15) is p-singular if and only if the point $A(\{v-1, p\})$ can be represented as a sum $X+Y$, $X\in K_1$, $Y\in K_2$ with integer coordinates, $r_1+r_2=n$, and R_i is given by (14). Lemma 3 implies that $K_1+K_2=K(T(\{-3, p\}), R=R_1+R_2)$.

Now, we calculate the point $M(\{m, p\})$ with maximal equal integer coordinates which is sum $X+Y$, $X\in K_1$, $Y\in K_2$ with integer coordinates. Let $S_i=(\{s_i, p\})\in\sigma_i$, $s_i>-1$, $S=(\{s, p\})\in\sigma$, $s>-2$, where σ_i and σ are the spheres with center T_i and radius R_i, $i=1, 2$, and T and R, respectively. Obviously $s_i=R_i/\sqrt{p}-3/2$ and $s=R/\sqrt{p}-3=s_1+s_2$.

Remark 1. Let $U((\{c, p\}), r) \subset \mathbb{R}^p$ be the closed ball with center $(\{c, p\})$ and radius r. Then $(z_1, ..., z_p) \in U$ if and only if $\sum_{j=1}^{p} z_j^2 \leqslant r^2 - pc^2 + 2c \sum_{j=1}^{p} z_j$. Suppose now that $\sum_{j=1}^{p} z_j$ is constant and denote the right-hand side of the last inequality by C. This means that in this case $(z_1, ..., z_p) \in U$ if and only if $\sum_{j=1}^{p} z_j^2 \leqslant C$.

Remark 2. It is clear that $\sum_{j=1}^{p} x_j^2$ takes its minimal value at the point $(\{k, p-\alpha\}, \ \{k+1, \alpha\})$ provided $x_1 \leqslant \cdots \leqslant x_p$, $\sum_{j=1}^{p} x_j = kp + \alpha$ is a constant, where x_j, k, α are integers.

Using Remark 1 and Remark 2 one can easily prove

LEMMA 4. *The point $M(\{m, p\})$ can be represented as a sum $X+Y$, $X \in K_1$, $Y \in K_2$ with integer coordinates if and only if there exist integers $(x_1, ..., x_p) \in K_1$, $(y_1, ..., y_p) \in K_2$, such that $\sum_{j=1}^{p} (x_j + y_j) = mp$.*

Let α_i be the maximal integer such that $0 \leqslant \alpha_i < p$, $X_i(\{[s_i], p-\alpha_i\}, \{[s_i]+1, \alpha_i\}) \in K_i$. For example, α_i can be found in the following way. The fact $X_i \in K_i$ implies that

$$(p-\alpha_i)([s_i]+\tfrac{3}{2})^2 + \alpha_i([s_i]+\tfrac{5}{2})^2 \leqslant R_i^2.$$

Therefore

$$\alpha_i = \left[\frac{R_i^2 - p([s_i]+3/2)^2}{2[s_i]+4}\right] = \left[\frac{R_i^2 - p([R_i/\sqrt{p} - 3/2] + 3/2)^2}{2[R_i/\sqrt{p} - 3/2] + 4}\right]. \qquad (21)$$

Note that X_1 and X_2 are the points with integer coordinates in K_1 and K_2, respectively, which have the maximal sums of their coordinates. Therefore by Lemma 4 we get the following corollary.

COROLLARY 4. *Let α_i be given by* (21). *Then the point $M = (\{m, p\})$,*

$$m = [s_1] + [s_2] + \left[\frac{\alpha_1 + \alpha_2}{p}\right] = \left[\frac{R_1}{\sqrt{p}} - \frac{3}{2}\right] + \left[\frac{R_2}{\sqrt{p}} - \frac{3}{2}\right] + \left[\frac{\alpha_1 + \alpha_2}{p}\right],$$

has maximal equal integer coordinates among the integer points $X+Y$, $X \in K_1$, $Y \in K_2$.

The next theorem gives necessary and sufficient condition for p-singularity.

THEOREM 3. *The matrix E of type* (15) *is p-singular if and only if there exists a natural number $r_1 \leqslant n-1$ such that*

$$\left[\frac{R_1}{\sqrt{p}} - \frac{3}{2}\right] + \left[\frac{R_2}{\sqrt{p}} - \frac{3}{2}\right] + \left[\frac{\alpha_1 + \alpha_2}{p}\right] \geqslant v - 1, \qquad (22)$$

where $r_2 = n - r_1$, $R_i = \sqrt{(r_i + 3/2)^2 + (p-9)/4}$ *and* α_i *is given by* (21), $i = 1, 2$.

Proof. If the matrix E of type (15) is p-singular then the assertion is obvious.

Now, suppose that there exists a natural number $r_1 \leqslant n-1$ and (22) holds. If $m = [s_1] + [s_2] + 1$ then $(\{[s_1], p - \alpha_1\}, \{[s_1] + 1, \alpha_1\}) + (\{[s_2] + 1, p - \alpha_1\}, \{[s_2], \alpha_1\}) = A(\{m, p\})$ and the claim holds. So, let us assume that $[s_1] + [s_2] \geqslant v - 1$. But $v \geqslant 0$ and therefore $-1 \leqslant v - 1 \leqslant [s_1] + [s_2]$. Let $X_k = (\{k-1, p\})$, $k = 0, ..., [s_1] + 1$ and $Y_l = (\{l-1, p\})$, $l = 0, ..., [s_2] + 1$. Clearly $X_k \in K_1$ and $Y_l \in K_2$. We have

$$\{X_k + Y_l : 0 \leqslant k \leqslant [s_1] + 1, 0 \leqslant l \leqslant [s_2] + 1\}$$
$$\supset \{(\{j, p\}) : -2 \leqslant j \leqslant [s_1] + [s_2]\}.$$

Thus the theorem is proved. ▌

Using Theorem 3, we describe all p-singular matrices of type (15). Consider separately the following two cases.

I. Let $p \leqslant 9$. Note that if $p = 4$ and $p = 9$ then Eq. (17) has no integer solutions $v \geqslant 0$, $n \geqslant 1$.

Now we show that if $p \neq 4$, $p \neq 9$, and $p \leqslant 9$ then all matrices of the type (15) are singular except for a few examples. To see this we wish to apply Theorem 3. Motivated by Theorem 2, a guess for the number r_1 comes from the smallest nonnegative integer solutions to

$$p\binom{\mu+2}{2} = \binom{r+2}{2} - 1. \tag{23}$$

Solving the quadratic equation for μ, we find that

$$\mu = \frac{-3p \pm \sqrt{p(4r^2 + 12r + p)}}{2p}.$$

For this to be a non-negative integer, we take the positive square root and need $4r^2 + 12r + p = q^2 p$, where q is an odd integer. Then $\mu = (-3 + q)/2$. Finally, we solve the quadratic $4r^2 + 12r = (q^2 - 1)p$ for its positive root: $r = (-3 + \sqrt{9 + (q^2 - 1)p})/2$. Since the equations for μ and r are monotonic in q, the desired solutions result from the smallest nonnegative odd integer q that produces integer solutions to both equations. The results are shown in Table I.

Write (23) in the form

$$p(2\mu + 3)^2 = (2r + 3)^2 + p - 9. \tag{24}$$

TABLE I

p	2	3	5	6	7	8
r	1	3	2	12	21	48
μ	0	1	0	4	7	16
D	1	3	1	6	7	8

We wish to consider Theorem 3 in case $r_1 = r$ and α_i is given by (21). Using (24) and (21), we calculate

$$\frac{R_1}{\sqrt{p}} - \frac{3}{2} = \frac{\sqrt{(2r+3)^2 + p - 9}}{2\sqrt{p}} - \frac{3}{2} = \frac{1}{2}(2\mu + 3) - \frac{3}{2} = \mu,$$

$$\alpha_1 = \left[\frac{(r+3/2)^2 + (p-9)/4 - p(\mu+3/2)^2}{2\mu+4}\right] = 0,$$

$[(\alpha_1 + \alpha_2)/p] = [\alpha_2/p] = 0$ because $0 \leqslant \alpha_2 < p$. Placing these numbers in (22), we find that Theorem 3 holds for $r_1 = r$ if and only if

$$r \leqslant n - 1 \qquad \text{and} \qquad \frac{R_2}{\sqrt{p}} \geqslant v - \mu + \frac{1}{2}. \tag{25}$$

Comparing (16) and (23), we find $r \leqslant n-1$ if and only if $v > \mu$. Using the equation for R_2 given in Theorem 3, by squaring the second equation we find that (25) is equivalent to $v > \mu$ and

$$(2n+3-2r)^2 + p - 9 \geqslant p(2v + 3 - 2\mu - 2)^2$$

or

$$\begin{aligned}(2n+3)^2 - 4r(2n+3) + 4r^2 + p - 9 \\ \geqslant p(2v+3)^2 - 4p(\mu+1)(2v+3) + 4p(\mu+1)^2.\end{aligned}$$

Using (17) this last inequality is equivalent to

$$(2n+3)r \leqslant p(\mu+1)(2v+3) + r^2 - p(\mu+1)^2 - 2. \tag{26}$$

Since $p(\mu+1)^2 + p(\mu+1) = r^2 + 3r$ by (24), we set

$$D := 3r - p(\mu+1) = p(\mu+1)^2 - r^2 \qquad \text{and} \qquad y = 2v + 3 \tag{27}$$

(the numbers D are computed in Table I). Therefore, (26) is equivalent to

$$(2n+3)r \leqslant p(\mu+1)\,y - (D+2), \tag{28}$$

which is a valid inequality since $p(\mu+1)y \geqslant 3p(\mu+1)(2(\mu+1)+3) > D+2$. Therefore, squaring (28) and making use of (17), we find that Theorem 3 applies with $r_1 = r$ if and only if $v > \mu$ and

$$h_p(y) := Dy^2 - 2(\mu+1)y(D+2) + ((D+2)^2 + (p-1)r^2)/p \geqslant 0 \quad (29)$$

for every $y \geqslant 2(\mu+1)+3$. In other words, all matrices of the type (15) with $v \geqslant \mu+1$ are p-singular if and only if (29) holds. It remains to show that (29) does hold. Since $D > 0$ (see Table I), $h_p(y)$ has its minimum value at

$$(\mu+1)\left(1+\frac{2}{D}\right) < 2(\mu+1)+3 =: y_0. \quad (30)$$

Therefore, $h_p(y)$ increases on $[y_0, \infty)$ and consequently, (29) holds if $h_p(y_0) \geqslant 0$. The latter can be verified by Table I.

We obtain from all these observations the following conclusions.

1. If $p=2$ all matrices E such that $v \geqslant 1$ are singular. Putting $p=2$ in (18) we get $x^2 - 2y^2 = -1$. It follows from [4] that this Diophantine equation has an infinite sequence of integer solutions which is derived from the particular solution (1, 1) by formulae (19). Obviously these solutions are odd and they are given by

$$\left| \begin{array}{l} x_m = a_m + 2b_m \\ y_m = a_m + b_m \end{array} \right. \qquad m \geqslant 1,$$

where

$$a_m = \frac{1}{2}((3+2\sqrt{2})^m + (3-2\sqrt{2})^m),$$

$$b_m = \frac{1}{2\sqrt{2}}((3+2\sqrt{2})^m - (3-2\sqrt{2})^m),$$

are the positive solutions of the equations of Pell $a^2 - 2b^2 = 1$. Consequently,

$$n_m = \frac{x_m - 3}{2}, \qquad v_m = \frac{y_m - 3}{2}, \qquad m \geqslant 1 \quad (31)$$

define all normal matrices in this case and *all these matrices are singular*. We list below some of them for small m:

$$m=1 \quad \left| \begin{array}{l} n_1 = 2 \\ v_1 = 1 \end{array} \right. \qquad m=2 \quad \left| \begin{array}{l} n_2 = 19 \\ v_2 = 13 \end{array} \right.$$

(the result (n_1, ν_1) was obtained by G. and R. Lorentz [2])

$$m=3 \quad \left| \begin{array}{l} n_3=118 \\ \nu_3=83 \end{array} \right. \qquad m=4 \quad \left| \begin{array}{l} n_4=695 \\ \nu_4=491. \end{array} \right.$$

2. If $p=3$ all matrices which satisfy $\nu \geqslant 2$ are singular. In this case (18) takes the form $x^2-3y^2=-2$. By [4] this equation has an infinite sequence of integer solutions which is derived from (1, 1) by formulae (19). These solutions are odd and they are given by

$$\left| \begin{array}{l} x_m=a_m+3b_m \\ y_m=a_m+b_m \end{array} \right., \qquad m \geqslant 1,$$

where

$$a_m=\frac{1}{2}((2+\sqrt{3})^m+(2-\sqrt{3})^m),$$

$$b_m=\frac{1}{2\sqrt{3}}((2+\sqrt{3})^m-(2-\sqrt{3})^m),$$

are the positive solutions of $a^2-2b^2=1$. All normal matrices are obtained from (31). *The matrices* (n_m, ν_m), $m \geqslant 2$ *are singular and evidently* (n_1, ν_1) *is almost regular.* We present here some of the small values of these parameters.

$$m=1 \quad \left| \begin{array}{l} n_1=1 \\ \nu_1=0 \end{array} \right. \qquad m=2 \quad \left| \begin{array}{l} n_2=8 \\ \nu_2=4 \end{array} \right.$$

$$m=3 \quad \left| \begin{array}{l} n_3=34 \\ \nu_3=19 \end{array} \right. \qquad m=4 \quad \left| \begin{array}{l} n_4=131 \\ \nu_4=75. \end{array} \right.$$

This case was considered indepenently by Le Mehaute [5].

3. If $p=5$ all matrices such that $\nu \geqslant 1$ are singular. The Diophantine equation $x^2-5y^2=-4$ has four sequences of solutions which are derived from $(1, 1)$, $(-1, 1)$, $(4, 2)$, $(-4, 2)$ but only the first and the second particular solutions yield odd solutions by (19). Consequently

$$\left| \begin{array}{l} x_m=a_m+5b_m \\ y_m=a_m+b_m \end{array} \right. \qquad \left| \begin{array}{l} x'_m=-a_m+5b_m \\ y'_m=a_m-b_m \end{array} \right., \qquad m \geqslant 1,$$

where

$$a_m=\frac{1}{2}((9+4\sqrt{5})^m+(9-4\sqrt{5})^m),$$

$$b_m=\frac{1}{2\sqrt{5}}((9+4\sqrt{5})^m-(9-4\sqrt{5})^m).$$

All normal matrices (n_m, v_m) and (n'_m, v'_m) are obtained from (x_m, y_m) and (x'_m, y'_m), $m \geqslant 1$, respectively by (31) and *all these matrices are singular*. We list below some of them for small m:

$$m=1 \quad \left| \begin{array}{l} n_1=13 \\ v_1=5 \end{array} \right. \quad \left| \begin{array}{l} n'_1=4 \\ v'_1=1 \end{array} \right. \qquad m=2 \quad \left| \begin{array}{l} n_2=259 \\ v_2=115 \end{array} \right. \quad \left| \begin{array}{l} n'_2=98 \\ v'_2=43 \end{array} \right.$$

((n'_1, v'_1) was obtained by G. and R. Lorentz [2])

$$m=3 \quad \left| \begin{array}{l} n_3=4673 \\ v_3=2089 \end{array} \right. \quad \left| \begin{array}{l} n'_3=1784 \\ v'_3=797. \end{array} \right.$$

4. If $p=6$ all matrices which satisfy $v \geqslant 5$ are singular. The equation $x^2-6y^2=-5$ has two sequences of odd solutions. All normal matrices are given by $n_m=\frac{1}{2}(a_m+6b_m-3)$, $v_m=\frac{1}{2}(a_m+b_m-3)$ and $n'_m=\frac{1}{2}(-a_m+6b_m-3)$, $v'_m=\frac{1}{2}(a_m-b_m-3)$, $m \geqslant 1$, where

$$a_m=\frac{1}{2}((5+2\sqrt{6})^m+(5-2\sqrt{6})^m),$$

$$b_m=\frac{1}{2\sqrt{6}}((5+2\sqrt{6})^m-(5-2\sqrt{6})^m).$$

The matrices (n_m, v_m) *and* (n'_m, v'_m), $m \geqslant 2$ *are singular*. It was proved in [2] that (n_1, v_1) *and* (n'_1, v'_1) *are almost regular*. We present here some of these matrices for small m:

$$m=1 \quad \left| \begin{array}{l} n_1=7 \\ v_1=2 \end{array} \right. \quad \left| \begin{array}{l} n'_1=2 \\ v'_1=0 \end{array} \right. \qquad m=2 \left| \begin{array}{l} n_2=83 \\ v_2=33 \end{array} \right. \quad \left| \begin{array}{l} n'_2=34 \\ v'_2=13 \end{array} \right.$$

$$m=3 \quad \left| \begin{array}{l} n_3=835 \\ v_3=340 \end{array} \right. \quad \left| \begin{array}{l} n'_3=350 \\ v'_3=142 \end{array} \right. \qquad m=4 \left| \begin{array}{l} n_4=8279 \\ v_4=3379 \end{array} \right. \quad \left| \begin{array}{l} n'_4=3478 \\ v'_4=1419. \end{array} \right.$$

5. If $p=7$ all matrices of type (15) such that $v \geqslant 8$ are singular. All normal matrices (n_m, v_m) and (n'_m, v'_m) are obtained from

$$\left| \begin{array}{l} x_m=a_m+7b_m \\ y_m=a_m+b_m \end{array} \right. \qquad \left| \begin{array}{l} x'_m=-a_m+7b_m \\ y'_m=a_m-b_m \end{array} \right., \qquad m \geqslant 1,$$

where

$$a_m=\frac{1}{2}((8+3\sqrt{7})^m+(8-3\sqrt{7})^m),$$

$$b_m=\frac{1}{2\sqrt{7}}((8+3\sqrt{7})^m-(8-3\sqrt{7})^m)$$

by formulae (31). *The matrices* (n_m, ν_m) *and* (n'_m, ν'_m), $m \geqslant 2$ *are singular.* G and R. Lorentz proved in [2] that (n'_1, ν'_1) *is almost regular*. It may be proved that (n_1, ν_1) *is almost regular*, too. A list of these matrices for small m is given below:

$$m=1 \quad \left| \begin{array}{l} n_1=13 \\ \nu_1=4 \end{array} \right. \quad \left| \begin{array}{l} n'_1=5 \\ \nu'_1=1 \end{array} \right. \qquad m=2 \left| \begin{array}{l} n_2=230 \\ \nu_2=86 \end{array} \right| \begin{array}{l} n'_2=103 \\ \nu'_2=38 \end{array}$$

$$m=3 \quad \left| \begin{array}{l} n_3=3688 \\ \nu_3=1393 \end{array} \right. \quad \left| \begin{array}{l} n'_3=1664 \\ \nu'_3=628. \end{array} \right.$$

6. If $p=8$ all matrices such that $\nu \geqslant 17$ are singular. All normal matrices (n_m, ν_m) and (n'_m, ν'_m) are obtained from $n_m = \frac{1}{2}(a_m + 8b_m - 3)$, $\nu_m = \frac{1}{2}(a_m + b_m - 3)$ and $n'_m = \frac{1}{2}(-a_m + 8b_m - 3)$, $\nu'_m = \frac{1}{2}(a_m - b_m - 3)$, $m = 2, 4, 6, \ldots$, where

$$a_m = \frac{1}{2}\left((3+\sqrt{8})^m + (3-\sqrt{8})^m\right),$$

$$b_m = \frac{1}{2\sqrt{8}}\left((3+\sqrt{8})^m - (3-\sqrt{8})^m\right).$$

It is easy to check that if m is odd then y_m is an even number. *The matrices* (n_m, ν_m) *and* (n'_m, ν'_m), $m \geqslant 4$ *are singular.* It may be proved that (n_2, ν_2) and (n'_2, ν'_2) *are almost regular*:

$$m=2 \quad \left| \begin{array}{l} n_2=31 \\ \nu_2=10 \end{array} \right. \quad \left| \begin{array}{l} n'_2=14 \\ \nu'_2=4 \end{array} \right. \qquad m=4 \left| \begin{array}{l} n_4=1103 \\ \nu_4=389 \end{array} \right| \begin{array}{l} n'_4=526 \\ \nu'_4=185 \end{array}$$

$$m=6 \quad \left| \begin{array}{l} n_6=37{,}519 \\ \nu_6=13{,}264 \end{array} \right. \quad \left| \begin{array}{l} n'_6=17{,}918 \\ \nu'_6=6334. \end{array} \right.$$

The above results show that the conjecture of G and R. Lorentz is not true (see [2]).

II. *Let* $p \geqslant 10$. We shall show that the necessary condition from Corollary 3 does not hold. In the case of equal multiplicities (13) takes the form

$$\sqrt{\left(r+\frac{3}{2}\right)^2 + \frac{p-9}{4}} + \sqrt{\left(n-r+\frac{3}{2}\right)^2 + \frac{p-9}{4}} \geqslant \sqrt{p}\,(\nu+2). \qquad (32)$$

LEMMA 5. *Let* $c \geqslant 2\beta_1 > 0$ *and* $\beta_1^2 + \alpha \geqslant 0$ *be real numbers. The function* $f(x) = \sqrt{x^2+\alpha} + \sqrt{(c-x)^2+\alpha}$, $x \in [\beta_1, c-\beta_1]$ *takes its maximal value: if* $\alpha > 0$ *at points* $x = \beta_1$, $x = c - \beta_1$; *if* $\alpha \leqslant 0$ *at point* $x = c/2$.

The proof is elementary. We omit it.

Applying Lemma 5 for $x=r+3/2$, $x\in[3/2, n+3/2]$, $\alpha=(p-9)/4>0$, and $c=n+3$ it follows that the expression of the left hand of the inequality (32) takes its maximal value when $r=0$. Using (17) we get

$$\max_{r\in[0,n]}\left\{\sqrt{\left(r+\frac{3}{2}\right)^2+\frac{p-9}{4}}+\sqrt{\left(n-r+\frac{3}{2}\right)^2+\frac{p-9}{4}}\right\}$$
$$=\frac{\sqrt{p}}{2}+\frac{1}{2}\sqrt{(2n+3)^2+p-9}<\frac{\sqrt{p}}{2}+\frac{\sqrt{p}}{2}(2v+3)=\sqrt{p}\,(v+2).$$

Consequently, we have proved that there does not exist a natural number $r\leqslant n-1$ which satisfies (32). Thus *if* $p\geqslant 10$ *there are no p-singular matrices.*

4. Hermitian Case of Arbitrary Multiplicities and Sufficient Conditions for Singularity

In this section we consider normal matrices of type (2) and the question for singularity will be treated again. Define a translation ϕ: $(\{z, p\})\to(\{z+3, p\})$. Thus $(\{-3, p\})\xrightarrow{\varphi}O(\{0, p\})$, $M(v_1-1, ..., v_p-1)\xrightarrow{\varphi}M'(v_1+2, ..., v_p+2)$, $K((\{-3, p\}), R_1+R_2)\xrightarrow{\varphi}K'((\{0, p\}), R_1+R_2)$. Let $K'_1(O, R_1)$, $K'_2(O, R_2)$, $K''_2(M', R_2)$ be the closed balls with centers O, O, M' and radii R_1, R_2, R_2, respectively.

Lemma 6. *The normal matrix E of type* (2) *is p-singular if and only if there exists a natural number* $r_1\leqslant[n/2]$ *such that there is a point of type* $(x_1+\frac{1}{2}, ..., x_p+\frac{1}{2})$ *in the* $K'_1\cap K''_2$, *where* x_j *are integers* $j=1, ..., p$, $r_2=n-r_1$ *and* R_i *is given by* (14).

Proof. Let E be a p-singular matrix. Then there exists a natural number $r_1\leqslant[n/2]$ such that $M=z_1+z_2$, $z_i\in K_i(T_i, R_i)$, $i=1, 2$. Denote by $K'_i=K_i+(\{\frac{3}{2}, p\})$, $z'_i=z_i+(\{\frac{3}{2}, p\})$. We show that $z'_1\in K'_1\cap K''_2$. Evidently, $z'_1\in K'_1$ and z'_1 is of type $(x_1+\frac{1}{2}, ..., x_p+\frac{1}{2})$. On the other hand $-z'_2\in K'_2$ and $K''_2=K'_2+M'$. According to this $-z'_2+M'\in K''_2$. But $z_1+z_2=M$ implies $z'_1+z'_2=M'$. Thus $-z'_2+M'=z'_1$.

Let us suppose that there exist $r_1\leqslant[n/2]$ such that $(x_1+\frac{1}{2}, ..., x_p+\frac{1}{2})\in K'_1\cap K''_2$. Obviously $(x_1+\frac{1}{2}, ..., x_p+\frac{1}{2})+Y(y_1+\frac{1}{2}, ..., y_p+\frac{1}{2})=M(v_1+2, ..., v_p+2)$, where $y_j=v_j-x_j+1$ are integers. $X\in K''_2$ implies $\sum_{j=1}^{p}(v_j-x_j+\frac{3}{2})^2\leqslant R_2^2$ and consequently $Y\in K'_1$. It remains to establish that $x_j\geqslant 0$, $y_j\geqslant 0$, $j=1, ..., p$. To this aim set

$$\begin{cases} x'_j=0, \quad y'_j=v_j+1 & \text{if} \quad x_j<0 \\ y'_j=0, \quad x'_j=v_j+1 & \text{if} \quad y_j<0 \\ x'_j=x_j, \quad y'_j=y_j & \text{if} \quad x_j\geqslant 0 \quad \text{and} \quad y_j\geqslant 0 \end{cases}, \qquad j=1, ..., p.$$

If $x_j < 0$ then $|x'_j + \frac{1}{2}| = \frac{1}{2} \leqslant |x_j + \frac{1}{2}|$, $y'_j \geqslant 1$ and $y'_j + \frac{1}{2} = y_j + x_j + \frac{1}{2} < y_j + \frac{1}{2}$. Similarly if $y_j < 0$ then $|x'_j + \frac{1}{2}| \leqslant |x_j + \frac{1}{2}|$ and $|y'_j + \frac{1}{2}| \leqslant |y_j + \frac{1}{2}|$. From Remark 1 and Remark 2 it follows that $X'(x'_1 + \frac{1}{2}, ..., x'_p + \frac{1}{2}) \in K'_1$ and $Y'(y'_1 + \frac{1}{2}, ..., y'_p + \frac{1}{2}) \in K'_2$. ∎

Lemma 6 gives us opportunity to check algorithmically the p-singularity of E. An effective algorithm can be written which checks if there is a point of type $(x_1 + \frac{1}{2}, ..., x_p + \frac{1}{2})$ in the section of two closed balls.

Henceforth it will be supposed that $v_1 \leqslant \cdots \leqslant v_p$. Now, using Lemma 6, we find sufficient conditions for singularity. When $r_1 = 1$ then $R_1 = \sqrt{p/4 + 4}$. Note that $\|(\{\frac{1}{2}, p-2\}, \frac{3}{2}, \frac{3}{2})\| = \sqrt{p/4 + 4} = R_1$, where $\|x\|$ is the Euclidean norm in $\mathbb{R}^p$ and $(\{\frac{1}{2}, p-1\}, \frac{5}{2}) \notin K'_1$. Consequently, E is p-singular if and only if $(\{\frac{1}{2}, p-2\}, \frac{3}{2}, \frac{3}{2}) \in K''_2$, i.e.,

$$\sum_{j=1}^{p-2} \left(v_j + \frac{3}{2}\right)^2 + \left(v_{p-1} + \frac{1}{2}\right)^2 + \left(v_p + \frac{1}{2}\right)^2 \leqslant R_2^2 = \left(n + \frac{1}{2}\right)^2 + \frac{p-9}{4}.$$

Using the normality of E, we write the above inequality in the form

$$v_{p-1} + v_p \geqslant n. \tag{33}$$

The sufficient condition (33) is very strong when p is a small number. Using (33) we completely characterize the Hermitian interpolation problem with arbitrary multiplicities in the case $p = 2, 3, 4$. This will be proved in the following simple geometrical way.

A. Let $p = 2$ *and* E *be a matrix of type* (2), *where* $0 \leqslant v_i \leqslant n-1$, $i = 1, 2$. The normality of E means that $|S_n| = |S_{v_1}| + |S_{v_2}|$. Suppose that $v_1 + v_2 \leqslant n - 1$, i.e., $(v_1 + 1) + (v_2 + 1) \leqslant n + 1$. This inequality shows that if we move S_{v_1} into the triangle with vertices $(n - v_1, 0)$, $(n, 0)$, $(n - v_1, v_1)$ (see Fig. 1) then S_{v_1} and S_{v_2} are disjoint sets. Obviously $(0, n) \notin S_{v_i}$, $i = 1, 2$.

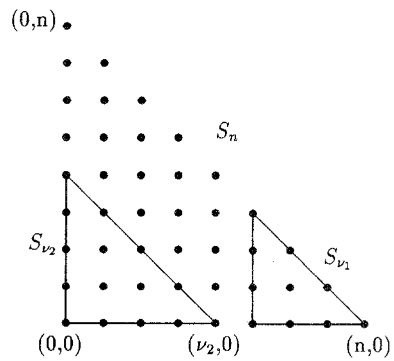

FIGURE 1

This is a contradiction with the normality of E. Thus $\nu_1+\nu_2 \geqslant n$ and we conclude that *if* $p=2$ *all matrices of type* (2) *are singular.*

This assertion can be proved directly. Let $z_i \in \mathbb{R}^2$, $i=1, 2$ be distinct arbitrary points. Consider the line $Q(x, y)=Ax+By+C$, where $z_1 \in Q$ and $z_2 \in Q$. Then $P(x, y)=Q(x, y)^n$ is a nontrivial polynomial, which has zeros of multiplicities ν_i+1 at z_i, $P(x, y) \in \Pi_n(\mathbb{R}^2)$, and our claim is proved.

Remark that the second proof holds for arbitrary Birkhoff matrices $E_i=(e_{i,j,k})_{(j,k)\in S}$ which satisfy $e_{i,j,k}=0$ if $j+k=n$, $i=1, 2$. Thus, we obtain the following necessary condition for almost regularity.

COROLLARY 5. *If* $E_1 \oplus E_2$ *is almost regular then there exist integers* $0 \leqslant j$, $k \leqslant n$, $j+k=n$ *which satisfy* $e_{1,j,k}+e_{2,j,k} \geqslant 1$.

But the next simple example shows that this condition is not sufficient.

EXAMPLE.

$$E=\begin{pmatrix} 0 & & \\ 0 & 0 & \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & & \\ 1 & 0 & \\ 1 & 0 & 0 \end{pmatrix}.$$

The matrix E is singular although it satisfies the conditions of Corollary 5. Moreover, it satisfies the Pólya condition, too (see [1]).

B. Let $p=3$ *and* E *be a matrix of type* (2). Suppose that $n \geqslant 2$ and therefore $\nu_3 \geqslant 1$. Assume that $\nu_2+\nu_3 \leqslant n-1$, i.e.,

$$(\nu_2+1)+(\nu_3+1) \leqslant n+1. \tag{34}$$

As in the previous case, we move S_{ν_2} into a triangle with vertices $(n-\nu_2, 0)$, $(n, 0)$, $(n-\nu_2, \nu_2)$ (see Fig. 2) and S_{ν_1} into the triangle with

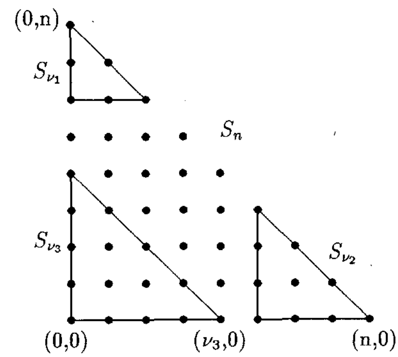

FIGURE 2

vertices $(0, n-\nu_1)$, $(0, n)$, $(\nu_1, n-\nu_1)$. It follows from (34) that S_{ν_i} are disjoint sets but $(\nu_3, \nu_3) \notin S_{\nu_i}$, $i=1, 2, 3$. Hence $|S_n| > |S_{\nu_1}| + |S_{\nu_2}| + |S_{\nu_3}|$ and a contradiction. Thus $\nu_2 + \nu_3 \geqslant n$ and we conclude that *if* $p=3$ *all matrices of type* (2) *are singular except the almost regular matrix* $n=1$, $\nu_1 = \nu_2 = \nu_3 = 0$.

C. Let $p=4$ *and E be matrix of type* (2). As in the previous case, we get that all normal matrices such that $\nu_3 + \nu_4 \leqslant n-1$ are given by $n = 2k+1$, $\nu_2 = \nu_3 = \nu_4 = k$, $\nu_1 = k-1$ and $n = 2k$, $\nu_4 = k$, $\nu_2 = \nu_3 = \nu_1 = k-1$. Using the method of shifts of G. and R. Lorentz, it may be proved that the above matrices are almost regular. So, if $p=4$, we conclude that *all matrices of type* (2) *are singular except matrices* $\nu_3 = \nu_2 = \nu_1 = k-1$, $\nu_4 = k$, $n = 2k$ *and* $\nu_4 = \nu_3 = \nu_2 = k$, $\nu_1 = k-1$, $n = 2k+1$, k *an arbitrary natural number, which are almost regular.*

Next, we find another sufficient condition for singularity when $p \geqslant 5$. If $r_1 = 2$ then $R_1 = \sqrt{p/4+10}$. But $\|(\{\frac{1}{2}, p-5\}, \{\frac{3}{2}, 5\})\| = R_1$. Consequently E is p-singular if $(\{\frac{1}{2}, p-5\}, \{\frac{3}{2}, 5\}) \in K_2''$. After some transformation, we obtain the sufficient condition

$$\nu_{p-4} + \nu_{p-3} + \nu_{p-2} + \nu_{p-1} + \nu_p \geqslant 2n-3. \tag{35}$$

Obviously, using the same method we may get many other sufficient conditions for singularity.

Our conjecture is that the set of singular matrices coincides with the set of p-singular matrices.

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